Goal: contraction mapping theorem

Appendix B

- Consider the equation

$$x = T(x)$$

where T is a map

- A solution x^* to this eq. is called "fixed point"

- Classical idea to find fixed point

start with initial guess x_0

Then $x_1 = T(x_0) \to x_2 = T(x_1) \to \cdots$

$$x_{k+1} = T(x_k)$$

successive approximation

- Theorem gives sufficient condition for existence of fixed point x^* and convergence $x_k \to x^*$

- Need to introduce $\underbrace{\text{Banach space}}_{\text{complete, normed, vec. space}}$

<u>Preliminaries:</u>

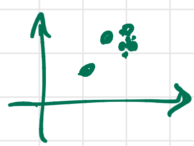
- Vector space $(\mathcal{X}, R)$ field, e.g $\mathbb{R}$, $\mathbb{C}$

  - $\forall x, y \in \mathcal{X}$, $\forall a, b \in \mathcal{X}$ $\Longrightarrow$ $ax + by \in \mathcal{X}$

- Norm: $\| \ \| : \mathcal{X} \to \mathbb{R}$

  * $\|x\| \geq 0$ with $\|x\| = 0$ iff $x = 0$
  * $\|\alpha x\| = |\alpha| \, \|x\|$
  * $\|x + y\| \leq \|x\| + \|y\|$ (triangle ineq)

- Convergence:



$x_k \to x$ as $k \to \infty$ if $\lim_{k \to \infty} \|x_k - x\| = 0$

$\forall \varepsilon > 0, \exists N$ s.t $\|x_k - x\| \leq \varepsilon \ \forall k > N$

- Closed set

  $S \subseteq X$ is closed iff $\forall$ convergent seq. in A has limit in A

- Cauchy seq. : $\{x_k\}$ is Cauchy if

  $$\lim_{m,n\to\infty} \|x_m - x_n\| = 0$$

  $\forall \varepsilon > 0, \exists N$ s.t $\|x_n - x_m\| \leq \varepsilon, \forall n,m > N$

- Every convergent seq. is Cauchy but not vice versa

- Space is complete if every Cauchy seq. converges

- Banach : vec space + norm + complete

Examples:

- $\mathbb{R}^n$ $\quad X = (x_1, \ldots, x_n)$

  $$\|X\|_p = (x_1^p + \cdots + x_n^p)^{\frac{1}{p}}$$

  it is complete.

- $C([a,b]; \mathbb{R}) = \{f : [a,b] \to \mathbb{R};\ f \text{ is cont.}\}$

  $$\|f\|_\infty = \max_{t \in [a,b]} |f(t)| \longrightarrow \text{complete}$$

  see pp. 654

Thm: Contraction mapping

- Let $\mathcal{X}$ be Banach, $S \subseteq \mathcal{X}$ closed subset
- T is a mapping from $\mathcal{X} \to \mathcal{X}$

T leaves S invariant

- if

(a) $T(x) \in S$ for all $x \in S$

(b) $\exists\, \rho \in [0,1)$ s.t. $\underbrace{\|T(x)-T(y)\| \leq \rho \|x-y\|}_{\text{contraction}}$

- Then,

(1) $\exists!\ x^* \in S$ s.t. $T(x^*) = x^*$

(2) x^* is obtained from succesive approx.

$$x_{k+1} = T(x_k) \implies x_k \to x^*$$

Proof:

- $\{x_k\}$ is Cauchy

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|T(x_k) - T(x_{k-1})\| \\
&\leq \rho \|x_k - x_{k-1}\| \\
&\leq \rho^2 \|x_{k-1} - x_{k-2}\| \\
&\ \ \vdots \\
&\leq \rho^k \|x_1 - x_0\|
\end{aligned}$$

- It follows that

$$\| x_{k+2} - x_k \| = \| x_{k+2} - x_{k+1} + x_{k+1} - x_k \|$$

triangle ineq. $\leq \| x_{k+2} - x_{k+1} \| + \| x_{k+1} - x_k \|$

$$\leq (\rho^{k+1} + \rho^k) \| x_1 - x_0 \|$$

- Therefore

$$\| x_{k+r} - x_k \| \leq \| x_{k+r} - x_{k+r-1} \| + \cdots + \| x_{k+1} - x_k \|$$

$$\leq (\rho^{k+r} + \rho^{k+r-1} + \cdots + \rho^k) \| x_1 - x_0 \|$$

$$= \frac{\rho^k - \rho^{k+r+1}}{1-\rho} \| x_1 - x_0 \|$$

$$\leq \frac{\rho^k}{1-\rho} \| x_1 - x_0 \|$$

$\longrightarrow 0$ as $k \uparrow \infty$

because $\rho < 1$

$\Longrightarrow \{x_k\}$ is Cauchy

- $\mathcal{X}$ is Banach $\rightarrow$ complete

$\rightarrow \exists\, x^* \in \mathcal{X}$ s.t $x_k \rightarrow x^*$

- S is closed, $x_k \in S \Longrightarrow x^* \in S$

- To show x^* is fixed point $x^* = T(x^*)$

$$
\begin{aligned}
\|x^* - T(x^*)\| &= \|x^* - x_k + x_k - T(x^*)\| \\
&\leq \|x^* - x_k\| + \|\underbrace{x_k}_{T(x_{k-1})} - T(x^*)\| \\
&\leq \|x^* - x_k\| + \rho \|x_{k-1} - x^*\| \\
&\rightarrow 0 \quad \text{as } k \uparrow \infty
\end{aligned}
$$

$$\Longleftrightarrow \|x^* - T(x^*)\| = 0 \Longrightarrow x^* = T(x^*)$$

- To show x^* is unique, suppose $y^* = T(y^*)$

$$
\begin{aligned}
\|x^* - y^*\| &= \|T(x^*) - T(y^*)\| \\
&\leq \rho \|x^* - y^*\| \quad \text{but } \rho < 1
\end{aligned}
$$

only possible if $\|x^* - y^*\| = 0 \Longrightarrow x^* = y^*$